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Analysis of a screw dislocation inside an elliptical inhomogeneity in piezoelectric solids

W. Deng, S. A. Meguid*

Engineering Mechanics and Design Laboratory, Department of Mechanical and Industrial Engineering, University of Toronto, 5 King's College Road, Toronto, Ontario, Canada M5S 3G8

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Abstract

This paper formulates and examines the electro-elastic coupling effects resulting from the presence of a screw dislocation inside an elliptical piezoelectric inhomogeneity embedded in an infinite piezoelectric matrix. The general solution to this problem is obtained by conformal mapping and Laurent series expansion of the corresponding complex potentials. The appropriate expressions of the field potentials and the field components are given explicitly in both the inhomogeneity and the surrounding matrix using a perturbation technique. The internal energy and the force on the dislocation are computed and several specific examples are provided to illustrate the validity and versatility of the developed formulations.

1. Introduction

Due to their favorable electro-mechanical behaviour, piezoelectric materials have been widely used as sensors and actuators. These devices are designed to work under combined electro-mechanical loads. The presence of various defects, such as dislocations, cracks and inclusions, can greatly influence their characteristics and coupled behaviour under load.

Significant progress has recently been made in the electro-elastic interaction caused by defects or inhomogeneities in piezoelectric materials. The works of McMeeking (1987), Pak (1990, 1992), Wang (1992), Suo et al. (1992), Chen (1993), Fan and Qin (1995), Zhang and Tong (1996), Sosa and Khutoryansky (1996), Zhong and Meguid (1997), Deng and Meguid (1997), Meguid and Deng (1997), among others, provide some recent contributions to the subject. Several basic results have been obtained, e.g., Deeg (1980) examined the effect of a dislocation, a crack and an inclusion upon the coupled response of piezoelectric solids. Pak (1990) derived closed-form solutions for a screw dislocation in an infinite piezoelectric solid, and showed the influence of the dislocation on

* Corresponding author. Tel.: 001 416 978 7753; fax: 001 416 978 5741; e-mail: meguid@me.utoronto.ca

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the electro-elastic coupling behaviour. Using the general eight-dimensional formalism, Suo et al. (1992) discussed the problem of interfacial cracks in bonded anisotropic piezoelectric media and obtained the solutions in terms of four analytical potential functions. More recently, Fan and Qin (1995) analyzed a piezoelectric ellipsoidal inhomogeneity embedded in a non-piezoelectric elastic matrix using the equivalent inclusion method. Zhang and Tong (1996) formulated the mechanical and electric fields around an elliptic cylindrical cavity in a piezoelectric material under remote antiplane shear and inplane electric fields by means of complex variable method. Meguid and Deng (1997) obtained the solution for the interaction problem of a dislocation outside an elliptical piezoelectric inhomogeneity in an infinite piezoelectric matrix. They found that when the inhomogeneity reduces to a cavity the electric field strength, both in the cavity and in the surrounding matrix, is not affected by the dislocation and is uniform inside the cavity.

It is the purpose of this paper to extend our previous work (Meguid and Deng, 1997) to investigate the electro-elastic coupling behaviour induced by a screw dislocation inside an elliptical piezoelectric inhomogeneity embedded in an unbounded piezoelectric matrix. The matrix is subjected to a remote antiplane shear and inplane electric field. The analysis is based upon the use of conformal mapping and the perturbation method. Following the introduction, Section 2 provides the basic field equations and the interfacial continuity conditions between the inhomogeneity and the matrix. In Section 3, a general series solution for the problem of a dislocation inside an elliptical inhomogeneity is derived explicitly. In Section 4, the total internal energy and the interaction energy between the dislocation and the inhomogeneity are considered and the force on the dislocation is computed. In Section 5, several examples are provided to illustrate the applications of the developed expression. The appropriate expressions for the field variables and field potentials, both in the inhomogeneity and in the matrix, are obtained for the following cases: (i) a circular piezoelectric inhomogeneity in a piezoelectric matrix; (ii) an elliptical elastic dielectric inhomogeneity in an elastic dielectric matrix; and (iii) an elliptical piezoelectric inhomogeneity in an elastic matrix. Finally, the paper is concluded in Section 6.

2. Basic equations

Let us consider an infinite piezoelectric medium containing an elliptical piezoelectric inhomogeneity and an isolated singularity, subject to the uniform remote mechanical and electric loads shown in Fig. 1. Both the inhomogeneity and the matrix are assumed to be transversely isotropic, while the singularity and the inhomogeneity are infinitely extended in a direction perpendicular to the x–y-plane. The inhomogeneity is assumed to be perfectly bonded with the matrix along the interface L and there are no concentrated forces and free charges lying on L. The singularity may be a line dislocation, a line force or a line charge. In our study, the singularity will be considered as a screw dislocation located at point (x_0, y_0) inside the inhomogeneity with the Burgers vector given as b_z. The regions occupied by the matrix and the inhomogeneity are referred to as Ω_1 and Ω_2, respectively.

For the present problem, only the anti-plane displacement w and the in-plane electric field E_x and E_y exist. They are independent of the longitudinal coordinate z, such that $w = w(x, y)$, $E_x = E_x(x, y)$ and $E_y = E_y(x, y)$. The respective governing field equations and the constitutive relations can be expressed as

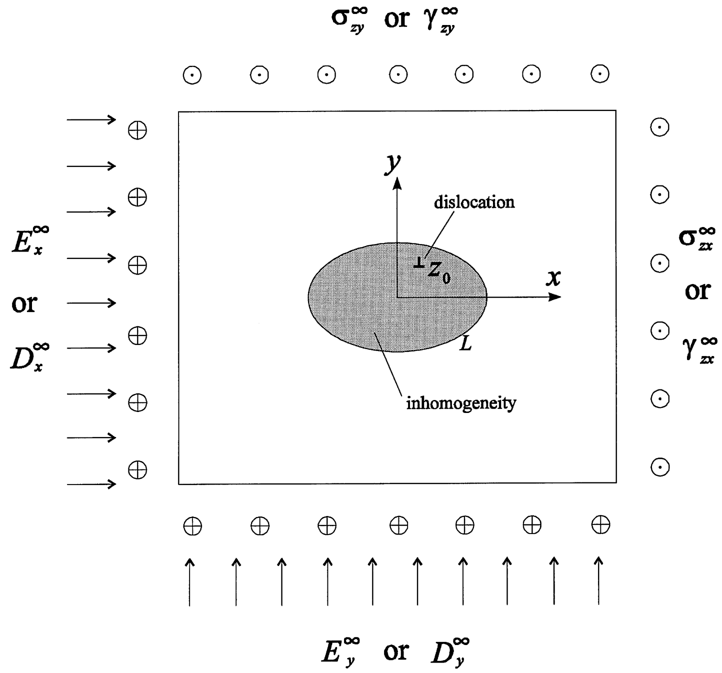

Fig. 1. A schematic of the electro-elastic interaction between a screw dislocation and an elliptical inhomogeneity in a piezoelectric material.

$$\frac{\partial \sigma_{zx}}{\partial x}+\frac{\partial \sigma_{zy}}{\partial y}=0 \tag{1}$$

$$\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}=0 \tag{2}$$

$$\sigma_{zx}=c_{44}\frac{\partial w}{\partial x}-e_{15}E_x,\quad \sigma_{zy}=c_{44}\frac{\partial w}{\partial y}-e_{15}E_y \tag{3}$$

$$D_x=e_{15}\frac{\partial w}{\partial x}+\varepsilon_{11}E_x,\quad D_y=e_{15}\frac{\partial w}{\partial y}+\varepsilon_{11}E_y \tag{4}$$

where σ_{zx} and σ_{zy} are the shear stresses, D_x and D_y are the electric displacements, and c_{44}, e_{15} and ε_{11} are the longitudinal shear modulus, piezoelectric modulus and dielectric modulus, respectively. Substituting (3) and (4) into (1) and (2) and noting that $E_i=-\phi_{,i}$, where $\phi(x,y)$ is the electric potential, we have

$$\begin{aligned} &c_{44}\nabla^2 w+e_{15}\nabla^2\phi=0\\ &e_{15}\nabla^2 w-\varepsilon_{11}\nabla^2\phi=0 \end{aligned} \tag{5}$$

where ∇^2 is the two-dimensional Laplacian operator. Let w and ϕ be the real parts of the analytic functions $\Psi(z)$ and $\Phi(z)$, such that:

$$w = \frac{1}{2c_{44}}[\Psi(z)+\overline{\Psi(z)}]$$

$$\phi = \frac{1}{2\varepsilon_{11}}[\Phi(z)+\overline{\Phi(z)}] \tag{6}$$

where $z = x+iy$ is the complex variable and the overbar refers to the complex conjugate. The expressions given in (6) satisfy (5) automatically. The electric field strength, the electric displacements and the stresses can be expressed by $\Psi(z)$ and $\Phi(z)$ as follows:

$$E_x - iE_y = -\frac{1}{\varepsilon_{11}}\Phi'(z), \quad D_x - iD_y = \frac{e_{15}}{c_{44}}\Psi'(z) - \Phi'(z),$$

$$\sigma_{zx} - i\sigma_{zy} = \Psi'(z) + \frac{e_{15}}{\varepsilon_{11}}\Phi'(z) \tag{7}$$

where prime denotes the derivatives with respect to the arguments. Using (7), the resultant force T and the resultant normal components S of the electric displacement along any arc AB can be calculated as

$$T = \int_A^B (\sigma_{zx}\,\mathrm{d}y - \sigma_{zy}\,\mathrm{d}x) = \frac{i}{2}\left\{[\overline{\Psi(z)} - \Psi(z)]_A^B + \frac{e_{15}}{\varepsilon_{11}}[\overline{\Phi(z)} - \Phi(z)]_A^B\right\}$$

$$S = \int_A^B (D_x\,\mathrm{d}y - D_y\,\mathrm{d}x) = \frac{i}{2}\left\{\frac{e_{15}}{c_{44}}[\overline{\Psi(z)} - \Psi(z)]_A^B - [\overline{\Phi(z)} - \Phi(z)]_A^B\right\} \tag{8}$$

where $[\,]_A^B$ represents the change in the bracketed function going from point A to B along the arc.

Let us now introduce the following mapping function

$$z = \Omega(\zeta) = \frac{c}{2}[R\zeta + (R\zeta)^{-1}], \quad R\zeta = \frac{1}{c}[z + (z^2 - c^2)^{1/2}] \tag{9}$$

with

$$\zeta = \xi + i\eta, \quad c = (a^2 - b^2)^{1/2} = a(1-\varepsilon^2)^{1/2}$$

$$R = \left(\frac{a+b}{a-b}\right)^{1/2} = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1/2}, \quad \varepsilon = \frac{b}{a} \tag{10}$$

where $2a$ and $2b$ are the major and minor diameters of the elliptical inhomogeneity. This mapping function transforms region Ω_1 of the z-plane into the exterior region of the unit circle Γ_1 ($\rho = 1$) in the transformed ζ-plane. It also transforms region Ω_2 into the annular region between the unit circle Γ_1 and a circle Γ_2 of radius $\rho = 1/R$ representing a cut from $-c$ to $+c$ in the z-plane, see Fig. 2. With the mapping function (9), eqns (6) and (8) can be rewritten in the ζ-plane as follows:

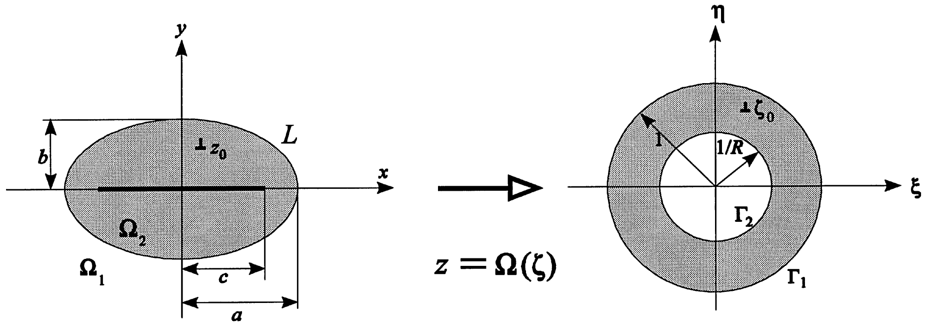

Fig. 2. A schematic of conformal mapping used.

$$w = \frac{1}{2c_{44}}[\Psi(\zeta)+\overline{\Psi(\zeta)}]$$

$$\phi = \frac{1}{2\varepsilon_{11}}[\Phi(\zeta)+\overline{\Phi(\zeta)}] \tag{11}$$

and

$$T = \frac{i}{2}\left\{[\overline{\Psi(\zeta)}-\Psi(\zeta)]_A^B+\frac{e_{15}}{\varepsilon_{11}}[\overline{\Phi(\zeta)}-\Phi(\zeta)]_A^B\right\}$$

$$S = \frac{i}{2}\left\{\frac{e_{15}}{c_{44}}[\overline{\Psi(\zeta)}-\Psi(\zeta)]_A^B-[\overline{\Phi(\zeta)}-\Phi(\zeta)]_A^B\right\} \tag{12}$$

where $\Psi(\zeta)$ and $\Phi(\zeta)$ imply $\Psi[\Omega(\zeta)]$ and $\Phi[\Omega(\zeta)]$, respectively. By extending the perturbation technique adopted by Stagni (1982) for isotropic elasticity, the general expressions for w and ϕ in (11) for the inhomogeneity can now be written as

$$\left.\begin{aligned} w_1 &= \frac{1}{2c_{44}^1}[\Psi_0(\zeta)+\overline{\Psi_0(\zeta)}+\Psi_1(\zeta)+\overline{\Psi_1(\zeta)}] \\ \phi_1 &= \frac{1}{2\varepsilon_{11}^1}[\Phi_0(\zeta)+\overline{\Phi_0(\zeta)}+\Phi_1(\zeta)+\overline{\Phi_1(\zeta)}] \end{aligned}\right\} \quad \zeta\in\Omega_1 \tag{13}$$

and

$$\left.\begin{aligned} w_2 &= \frac{1}{2c_{44}^2}[\Psi_*(\zeta)+\overline{\Psi_*(\zeta)}+\Psi_2(\zeta)+\overline{\Psi_2(\zeta)}] \\ \phi_2 &= \frac{1}{2\varepsilon_{11}^2}[\Phi_*(\zeta)+\overline{\Phi_*(\zeta)}+\Phi_2(\zeta)+\overline{\Phi_2(\zeta)}] \end{aligned}\right\} \quad \zeta\in\Omega_2 \tag{14}$$

where the subscripts (or superscripts) 1 and 2 represent the matrix Ω_1 and the inhomogeneity Ω_2, respectively. The functions Ψ_0 and Φ_0 (Ψ_* and Φ_*) are the field potentials which are holomorphic in $\Omega_1(\Omega_2)$, except at some singular points such as those located at dislocations and concentrated forces or charges. The functions Ψ_1 and Φ_1 (or Ψ_2 and Φ_2) are the field potentials which are holomorphic in region Ω_1 (or Ω_2).

The assumption of perfect bonding and that of no free charges and forces along the interface between regions Ω_1 and Ω_2 imply the continuity of displacement, electric potential, traction and normal components of the electric displacement across the elliptical interface. These conditions can be expressed as

$$w_1 = w_2, \quad \phi_1 = \phi_2, \quad T_1 = T_2, \quad S_1 = S_2 \quad \text{on } \Gamma_1 (\zeta = \sigma = e^{i\vartheta}). \tag{15}$$

Substituting (13) and (14) into (15) yields

$$\mu_1[\Psi_0(\sigma)+\overline{\Psi_0(\sigma)}+\Psi_1(\sigma)+\overline{\Psi_1(\sigma)}] = \Psi_*(\sigma)+\overline{\Psi_*(\sigma)}+\Psi_2(\sigma)+\overline{\Psi_2(\sigma)} \tag{16a}$$

$$\mu_2[\Phi_0(\sigma)+\overline{\Phi_0(\sigma)}+\Phi_1(\sigma)+\overline{\Phi_1(\sigma)}] = \Phi_2(\sigma)+\overline{\Phi_2(\sigma)} \tag{16b}$$

$$\begin{aligned}&[\overline{\Psi_0(\sigma)}-\Psi_0(\sigma)+\overline{\Psi_1(\sigma)}-\Psi_1(\sigma)]+\alpha_1[\overline{\Phi_0(\sigma)}-\Phi_0(\sigma)+\overline{\Phi_1(\sigma)}-\Phi_1(\sigma)]\\ &\qquad = [\overline{\Psi_*(\sigma)}-\Psi_*(\sigma)+\overline{\Psi_2(\sigma)}-\Psi_2(\sigma)]+\alpha_2[\overline{\Phi_2(\sigma)}-\Phi_2(\sigma)]\end{aligned} \tag{16c}$$

$$\begin{aligned}&\beta_1[\overline{\Psi_0(\sigma)}-\Psi_0(\sigma)+\overline{\Psi_1(\sigma)}-\Psi_1(\sigma)]-[\overline{\Phi_0(\sigma)}-\Phi_0(\sigma)+\overline{\Phi_1(\sigma)}-\Phi_1(\sigma)]\\ &\qquad = \beta_2[\overline{\Psi_*(\sigma)}-\Psi_*(\sigma)+\overline{\Psi_2(\sigma)}-\Psi_2(\sigma)]-[\overline{\Phi_2(\sigma)}-\Phi_2(\sigma)]\end{aligned} \tag{16d}$$

where

$$\begin{aligned}&\mu_1 = c_{44}^2/c_{44}^1, \quad \mu_2 = \varepsilon_{11}^2/\varepsilon_{11}^1, \quad \alpha_1 = e_{15}^1/\varepsilon_{11}^1, \quad \alpha_2 = e_{15}^2/\varepsilon_{11}^2\\ &\beta_1 = e_{15}^1/c_{44}^1, \quad \beta_2 = e_{15}^2/c_{44}^2.\end{aligned} \tag{17}$$

In addition, the following conditions must be satisfied on Γ_2

$$\Psi_2(\sigma/R) = \Psi_2(\bar{\sigma}/R), \quad \Phi_2(\sigma/R) = \Phi_2(\bar{\sigma}/R) \tag{18}$$

since the points σ/R and $\bar{\sigma}/R$ correspond to the same points of the cut from $-c$ to $+c$ in the z-plane.

Our task now is to determine the complex potentials Ψ_j and Φ_j ($j = 1, 2$) for regions Ω_1 and Ω_2 which satisfy conditions (16) and (18).

3. General solutions

When a dislocation is located at the point $z = z_0 = \Omega(\zeta_0)$ inside the inhomogeneity, care should be taken in choosing Ψ_0 and Φ_0 (Ψ_* and Φ_*) since they are multi-valued in $\Omega_1(\Omega_2)$. After considering the singularity and multi-valued behaviour caused by the dislocation, an appropriate choice of the field potentials Ψ_0, Φ_0, Ψ_* and Φ_* are made as follows

$$\left.\begin{aligned}\Psi_0(\zeta) &= h_1 \ln\zeta + p_0\Omega(\zeta)\\ \Phi_0(\zeta) &= h_2 \ln\zeta + q_0\Omega(\zeta)\end{aligned}\right\} \quad \zeta\in\Omega_1 \tag{19}$$

$$\left.\begin{aligned} \Psi_*(\zeta) &= \frac{c_{44}^2 b_z}{2\pi i}\ln[\Omega(\zeta)-\Omega(\zeta_0)] \\ \Phi_*(\zeta) &= 0 \end{aligned}\right\} \quad \zeta \in \Omega_2 \tag{20}$$

where the Burgers vector b_z is a real number, h_1 and h_2 are unknown complex constants which will be determined from the interface continuity conditions. p_0 and q_0 are complex constants which can be determined from the mechanical and electric loading conditions at infinity and can thus be taken as the remote equivalent mechanical and electric fields, respectively. There are four possible combinations of remote mechanical and electric loadings:

Case 1: remote mechanical strains γ_{zx}^∞, γ_{zy}^∞ and remote electric field strength E_x^∞ and E_y^∞;
Case 2: remote mechanical stresses σ_{zx}^∞, σ_{zy}^∞ and remote electric displacements D_x^∞ and D_y^∞;
Case 3: remote mechanical strains γ_{zx}^∞, γ_{zy}^∞ and remote electric displacements D_x^∞ and D_y^∞; and
Case 4: remote mechanical stresses σ_{zx}^∞, σ_{zy}^∞ and remote electric field strength E_x^∞ and E_y^∞.

Each case corresponds to a pair of p_0 and q_0, which are provided in Appendix 1.

With the aid of the mapping function (9) and the following relation

$$\ln(1-\zeta) = -\sum_{k=1}^{\infty}\frac{\zeta^{k+1}}{k+1} \quad |\zeta| < 1 \tag{21}$$

eqns (19) and (20) can be expressed by Laurent's expansions, as follows:

$$\left.\begin{aligned} \Psi_0(\zeta) &= h_1 \ln\zeta + \sum_{k=0}^{\infty}[a_k^0\zeta^{k+1} + b_k^0\zeta^{-(k+1)}] \\ \Phi_0(\zeta) &= h_2 \ln\zeta + \sum_{k=0}^{\infty}[c_k^0\zeta^{k+1} + d_k^0\zeta^{-(k+1)}] \end{aligned}\right\} \quad \zeta \in \Omega_1 \tag{22}$$

$$\left.\begin{aligned} \Psi_*(\zeta) &= \frac{c_{44}^2 b_z}{2\pi i}\ln\zeta + \sum_{k=0}^{\infty} b_k^*\zeta^{-(k+1)} \\ \Phi_*(\zeta) &= 0 \end{aligned}\right\} \quad \frac{1}{R} \leqslant |\zeta_0| < |\zeta| \leqslant 1 \tag{23}$$

where the constant terms denoting the equipotential field and the translation of a rigid body have been omitted. The coefficients a_k^0, b_k^0, c_k^0, d_k^0 and b_k^* are given as follows:

$$a_k^0 = \begin{cases} p_0 cR/2 & k = 0 \\ 0 & k = 1, 2, \ldots \end{cases} \tag{24a}$$

$$b_k^0 = \begin{cases} p_0 c/2R & k = 0 \\ 0 & k = 1, 2, \ldots \end{cases} \tag{24b}$$

$$c_k^0 = \begin{cases} q_0 cR/2 & k = 0 \\ 0 & k = 1, 2, \ldots \end{cases} \tag{24c}$$

$$d_k^0 = \begin{cases} q_0 c/2R & k = 0 \\ 0 & k = 1, 2, \ldots \end{cases} \tag{24d}$$

$$b_k^* = \frac{c_{44}^2 b_z}{2\pi i}\left(-\frac{1}{k+1}\right)[\zeta_0^{k+1} + (R^2\zeta_0)^{-(k+1)}] \quad k = 0, 1, 2, \ldots. \tag{24e}$$

Noting that in the ζ-plane, $\Psi_1(\zeta)$ and $\Phi_1(\zeta)$ are holomorphic in the exterior of the unit circle Γ_1 and $\Psi_2(\zeta)$ and $\Phi_2(\zeta)$ are holomorphic in the annular region between the unit circle Γ_1 and the circle Γ_2 of radius $\rho = 1/R$ (Fig. 2), they can be expressed by the following Laurent's expansions:

$$\Psi_1(\zeta) = \sum_{k=0}^{\infty} b_k^1 \zeta^{-(k+1)}, \quad \Phi_1(\zeta) = \sum_{k=0}^{\infty} d_k^1 \zeta^{-(k+1)} \quad \zeta \in \Omega_1 \tag{25}$$

$$\left.\begin{aligned} \Psi_2(\zeta) &= \sum_{k=0}^{\infty} [a_k^2 \zeta^{k+1} + b_k^2 \zeta^{-(k+1)}] \\ \Phi_2(\zeta) &= \sum_{k=0}^{\infty} [c_k^2 \zeta^{k+1} + d_k^2 \zeta^{-(k+1)}] \end{aligned}\right\} \quad \zeta \in \Omega_2. \tag{26}$$

Substituting (26) into (18) yields

$$a_k^2 = R^{2(k+1)} b_k^2, \quad c_k^2 = R^{2(k+1)} d_k^2. \tag{27}$$

With relation (27), equation (26) reduces to

$$\left.\begin{aligned} \Psi_2(\zeta) &= \sum_{k=0}^{\infty} [a_k^2 \zeta^{k+1} + a_k^2 R^{-2(k+1)} \zeta^{-(k+1)}] \\ \Phi_2(\zeta) &= \sum_{k=0}^{\infty} [c_k^2 \zeta^{k+1} + c_k^2 R^{-2(k+1)} \zeta^{-(k+1)}] \end{aligned}\right\} \quad \zeta \in \Omega_2. \tag{28}$$

Substituting expressions (22), (23), (25) and (28) into the continuity condition (16), and noting that on the unit circle Γ_1 of Fig. 2, $\zeta = \sigma = 1/\bar{\sigma}$, we have

$$\mu_1 b_k^1 = -\mu_1(\bar{a}_k^0 + b_k^0) + b_k^* + \bar{a}_k^2 + a_k^2 R^{-2(k+1)} \tag{29a}$$

$$\mu_2 d_k^1 = -\mu_2(\bar{c}_k^0 + d_k^0) + \bar{c}_k^2 + c_k^2 R^{-2(k+1)} \tag{29b}$$

$$b_k^1 + \alpha_1 d_k^1 = (\bar{a}_k^0 - b_k^0 + b_k^*) + \alpha_1(\bar{c}_k^0 - d_k^0) + (a_k^2 + \alpha_2 c_k^2) R^{-2(k+1)} - (\bar{a}_k^2 + \alpha_2 \bar{c}_k^2) \tag{29c}$$

$$\beta_1 b_k^1 - d_k^1 = \beta_1(\bar{a}_k^0 - b_k^0) - (\bar{c}_k^0 - d_k^0) + \beta_2 b_k^* + (\beta_2 a_k^2 - c_k^2) R^{-2(k+1)} - (\beta_2 \bar{a}_k^2 - \bar{c}_k^2) \tag{29d}$$

and

$$h_1 = \frac{c_{44}^1 b_z}{2\pi i}, \quad h_2 = 0. \tag{30}$$

For a given k ($k = 0, 1, 2, \ldots$), eqns (29a)–(29d) provide a system of four linear equations with four unknowns a_k^2, b_k^1, c_k^2 and d_k^1. These unknown coefficients can be solved and expressed in terms of the specified coefficients a_k^0, b_k^0, c_k^0, d_k^0 and b_k^* as being:

$$a_k^2 = I_k^{(1)} a_k^0 + J_k^{(1)} \bar{a}_k^0 + L_k^{(1)} c_k^0 + N_k^{(1)} \bar{c}_k^0 + U_k^{(1)} b_k^* + V_k^{(1)} \bar{b}_k^* \tag{31a}$$

$$b_k^1 = I_k^{(2)} a_k^0 + J_k^{(2)} \bar{a}_k^0 + L_k^{(2)} c_k^0 + N_k^{(2)} \bar{c}_k^0 + U_k^{(2)} b_k^* + V_k^{(2)} \bar{b}_k^* - b_k^0 \tag{31b}$$

$$c_k^2 = I_k^{(3)} a_k^0 + J_k^{(3)} \bar{a}_k^0 + L_k^{(3)} c_k^0 + N_k^{(3)} \bar{c}_k^0 + U_k^{(3)} b_k^* + V_k^{(3)} \bar{b}_k^* \tag{31c}$$

$$d_k^1 = I_k^{(4)} a_k^0 + J_k^{(4)} \bar{a}_k^0 + L_k^{(4)} c_k^0 + N_k^{(4)} \bar{c}_k^0 + U_k^{(4)} b_k^* + V_k^{(4)} \bar{b}_k^* - d_k^0 \tag{31d}$$

where the coefficients $I_k^{(n)}$, $J_k^{(n)}$, $L_k^{(n)}$, $N_k^{(n)}$, $U_k^{(n)}$ and $V_k^{(n)}$ $(n = 1, 2, 3, 4)$ are provided in Appendix 2.

Substituting (24) into (31), all the coefficients in the series expansions (25) and (28) for $\Psi_1(\zeta)$, $\Phi_1(\zeta)$, $\Psi_2(\zeta)$ and $\Phi_2(\zeta)$ are determined. Furthermore, with the help of the following relation

$$\frac{\mathrm{d}\xi}{\partial z} = \frac{2R\zeta^2}{c(R^2\zeta^2 - 1)} \tag{32}$$

the electric field strength, electric displacements and the stresses can be calculated from (7), as

$$E_{x1} - iE_{y1} = -\frac{1}{\varepsilon_{11}^1}\left[q_0 - \frac{2R\zeta^2}{c(R^2\zeta^2-1)}\sum_{k=0}^{\infty}(k+1)d_k^1\zeta^{-(k+2)}\right] \tag{33a}$$

$$D_{x1} - iD_{y1} = \frac{e_{15}^1}{c_{44}^1}\left\{p_0 + \frac{2R\zeta^2}{c(R^2\zeta^2-1)}\left[\frac{c_{44}^1 b_z}{2\pi i}\frac{1}{\zeta} - \sum_{k=0}^{\infty}(k+1)b_k^1\zeta^{-(k+2)}\right]\right\} - \left[q_0 - \frac{2R\zeta^2}{c(R^2\zeta^2-1)}\sum_{k=0}^{\infty}(k+1)d_k^1\zeta^{-(k+2)}\right] \tag{33b}$$

$$\sigma_{zx1} - i\sigma_{zy1} = p_0 + \frac{2R\zeta^2}{c(R^2\zeta^2-1)}\left[\frac{c_{44}^1 b_z}{2\pi i}\frac{1}{\zeta} - \sum_{k=0}^{\infty}(k+1)b_k^1\zeta^{-(k+2)}\right] + \frac{e_{15}^1}{\varepsilon_{11}^1}\left[q_0 - \frac{2R\zeta^2}{c(R^2\zeta^2-1)}\sum_{k=0}^{\infty}(k+1)d_k^1\zeta^{-(k+2)}\right] \tag{33c}$$

in the matrix, and

$$E_{x2} - iE_{y2} = -\frac{1}{\varepsilon_{11}^2}\frac{2R\zeta^2}{c(R^2\zeta^2-1)}\sum_{k=0}^{\infty}(k+1)c_k^2[\zeta^k - R^{-2(k+1)}\zeta^{-(k+2)}] \tag{34a}$$

$$D_{x2} - iD_{y2} = \frac{e_{15}^2}{c_{44}^2}\left\{\frac{c_{44}^2 b_z}{2\pi i}\frac{1}{z-z_0} + \frac{2R\zeta^2}{c(R^2\zeta^2-1)}\sum_{k=0}^{\infty}(k+1)a_k^2[\zeta^k - R^{-2(k+1)}\zeta^{-(k+2)}]\right\} - \frac{2R\zeta^2}{c(R^2\zeta^2-1)}\sum_{k=0}^{\infty}(k+1)c_k^2[\zeta^k - R^{-2(k+1)}\zeta^{-(k+2)}] \tag{34b}$$

$$\sigma_{zx2}-i\sigma_{zy2} = \frac{c_{44}^2 b_z}{2\pi i}\frac{1}{z-z_0}+\frac{2R\zeta^2}{c(R^2\zeta^2-1)}\sum_{k=0}^{\infty}(k+1)a_k^2[\zeta^k-R^{-2(k+1)}\zeta^{-(k+2)}]$$
$$+\frac{e_{15}^e}{\varepsilon_{11}^2}\frac{2R\zeta^2}{c(R^2\zeta^2-1)}\sum_{k=0}^{\infty}(k+1)c_k^2[\zeta^k-R^{-2(k+1)}\zeta^{-(k+2)}] \quad (34c)$$

in the inhomogeneity. The problem is thus solved. It should be pointed out that the current problem can be divided into two independent problems: one is a dislocation inside a piezoelectric inhomogeneity in an infinite matrix without the remote loading; the other is a piezoelectric inhomogeneity embedded in a piezoelectric matrix under remote antiplane shear and inplane electric field. These two problems are uncoupled due to the linearity of the governing field equations and constitutive relations (1)–(4). They can be discussed separately and superposed to result in the field components as given in (33) and (34).

The field potentials for the later problem can then be given in closed forms in the physical z-plane as:

$$\Psi_0(z)+\Psi_1(z) = p_0 z+\tfrac{1}{2}[(I_0^{(2)}R^2-1)p_0+J_0^{(2)}R^2\bar{p}_0$$
$$+L_0^{(2)}R^2 q_0+N_0^{(2)}R^2\bar{q}_0][z-(z^2-c^2)^{1/2}] \quad z\in\Omega_1 \quad (35a)$$

$$\Phi_0(z)+\Phi_1(z) = q_0 z+\tfrac{1}{2}[I_0^{(4)}R^2 p_0+J_0^{(4)}R^2\bar{p}_0$$
$$+(L_0^{(4)}R^2-1)q_0+N_0^{(4)}R^2\bar{q}_0][z-(z^2-c^2)^{1/2}] \quad z\in\Omega_1 \quad (35b)$$

$$\Psi_*(z)+\Psi_2(z) = [I_0^{(1)}p_0+J_0^{(1)}\bar{p}_0+L_0^{(1)}q_0+N_0^{(1)}\bar{q}_0]z \quad z\in\Omega_2 \quad (36a)$$

$$\Phi_*(z)+\Phi_2(z) = [I_0^{(3)}p_0+J_0^{(3)}\bar{p}_0+L_0^{(3)}q_0+N_0^{(3)}\bar{q}_0]z \quad z\in\Omega_2. \quad (36b)$$

Substitution of the above solutions into (7) produces the field components. It can be easily found that the stress, the electric field strength and the electric displacement inside the elliptical inhomogeneity are uniform, since $\Psi_2(z)$ and $\Phi_2(z)$ are linear functions of z. The solutions (35) and (36) are in agreement with those derived by Zhong and Meguid (1996).

4. Integral energy and force on the dislocation

One of the major interests in discussing dislocation problems is the interaction energy and force on dislocations. In the present problem, the total internal energy W due to a dislocation located at a point z_0, in the absence of the remote mechanical and electric fields, is equal to the work required to produce the dislocation, i.e.,

$$W = \tfrac{1}{2}b_z T$$

where T is the resultant force along the dislocation line running through regions 1 and 2. It follows from (8) or (12) that

$$T = \frac{i}{2}\{[\overline{\Psi_0(z)}+\overline{\Psi_1(z)}-\Psi_0(z)-\Psi_1(z)]+\alpha_1[\overline{\Phi_0(z)}+\overline{\Phi_1(z)}-\Phi_0(z)-\Phi_1(z)]\}_{z\to\Lambda}^{z\to\tilde{z}\in L}$$

$$+\frac{i}{2}\{[\overline{\Psi_*(z)}+\overline{\Psi_2(z)}-\Psi_*(z)-\Psi_2(z)]+\alpha_2[\overline{\Phi_*(z)}+\overline{\Phi_2(z)}-\Phi_*(z)-\Phi_2(z)]\}_{z\to\tilde{z}\in L}^{z\to z_0+\delta} \quad (37)$$

where Λ is a large constant, δ is a small number representing the dislocation core radius.

Substituting (19), (20), (25), (26) and (30) into (37) and noting that the value of the terms in (37) cancel each other at the point $z = \tilde{z}$ on the interface L, we obtain the following expression for the total internal energy

$$W = \frac{1}{2}b_z\,\mathrm{Im}\left\{\frac{c_{44}^2 b_z}{2\pi i}\ln\delta - \frac{c_{44}^1 b_z}{2\pi i}\ln\Lambda + \mathrm{const}\right\} + \delta W \quad (38)$$

where Im stands for the imaginary part of the complex expressions given above. δW is the electro-elastic interaction energy between the dislocation and the piezoelectric inhomogeneity, which can be obtained by excluding the dislocation singularity as

$$\delta W = \frac{1}{2}b_z\,\mathrm{Im}\left\{\sum_{k=0}^{\infty}(a_k^2+\alpha_2 c_k^2)[\zeta_0^{k+1}+R^{-2(k+1)}\zeta_0^{-(k+1)}]\right\} \quad (39)$$

where ζ_0 is related to $z_0 = (c/2)[R\zeta_0 + (1/R\zeta_0)]$. For a dislocation lying at the point $z_0 = x_0$ on the x-axis in the inhomogeneity, the interaction energy becomes

$$\delta W = \frac{c_{44}^2 b_z^2}{4\pi}\left\{\sum_{k=0}^{\infty}\frac{1}{k+1}[U_k^{(1)}-V_k^{(1)}+\alpha_2(U_k^{(3)}-V_k^{(3)})][\zeta_0^{k+1}+R^{-2(k+1)}\zeta_0^{-(k+1)}]^2\right\}. \quad (40)$$

The force F on the dislocation at the point x_0 is defined as the negative gradient with respect to x_0 of the interaction energy δW, i.e.,

$$F = -\frac{\partial(\delta W)}{\partial x_0}$$

which leads to

$$F = -\frac{c_{44}^2 b_z^2}{2\pi}\sum_{k=0}^{\infty}\left\{\frac{2R\zeta_0[U_k^{(1)}-V_k^{(1)}+\alpha_2(U_k^{(3)}-V_k^{(3)})]}{c(R^2\zeta_0^2-1)}[\zeta_0^{2(k+1)}-(R^2\zeta_0)^{-2(k+1)}]\right\}. \quad (41)$$

The above expression includes the electro-mechanical coupling effects.

5. Examples

In some special cases, the series solutions (25) and (28) for the general electro-elastic coupling problem between a screw dislocation and an elliptical piezoelectric inhomogeneity can be reduced to simpler closed-form expressions. In this section, several special examples are provided to illustrate the versatility of the general solutions. Three combinations of the inhomogeneity and the matrix will be considered. They are: (i) a circular piezoelectric inhomogeneity in a piezoelectric matrix; (ii) an elliptical elastic dielectric inhomogeneity in an elastic dielectric matrix; and (iii) an elliptical piezoelectric inhomogeneity in an elastic matrix.

5.1. *Circular piezoelectric inhomogeneity in piezoelectric matrix*

When a screw dislocation is located inside a circular piezoelectric inhomogeneity ($a = b$), the mapping function (9) becomes $z = \Omega(\zeta) = a\zeta$. Using relations (21) and (31), the field potentials Ψ_1, Φ_1, Ψ_2 and Φ_2 in (25) and (28) can be obtained in closed forms and the solutions in Ω_1 and Ω_2 are given by

$$\Psi_0(z)+\Psi_1(z) = \frac{c_{44}^1 b_z}{2\pi i}\ln\frac{z}{a}+\frac{c_{44}^2 b_z}{2\pi i}\Delta_{10}\ln\left(1-\frac{z_0}{z}\right)+p_0 z+(\bar{p}_0\Delta_3+\bar{q}_0\Delta_4)\frac{a^2}{z} \quad z\in\Omega_1 \tag{42a}$$

$$\Phi_0(z)+\Phi_1(z) = \frac{c_{44}^2 b_z}{2\pi i}\Delta_{11}\ln\left(1-\frac{z_0}{z}\right)+q_0 z+(\bar{p}_0\Delta_7+\bar{q}_0\Delta_8)\frac{a^2}{z} \quad z\in\Omega_1 \tag{42b}$$

$$\Psi_*(z)+\Psi_2(z) = \frac{c_{44}^2 b_z}{2\pi i}\left[\ln(z-z_0)+\Delta_3\ln\left(1-\frac{\bar{z}_0 z}{a^2}\right)\right]+(p_0\Delta_1+q_0\Delta_2)z \quad z\in\Omega_2 \tag{43a}$$

$$\Phi_*(z)+\Phi_2(z) = -\frac{c_{44}^2 b_z}{2\pi i}\Delta_9\ln\left(1-\frac{\bar{z}_0 z}{a^2}\right)+(p_0\Delta_5+q_0\Delta_6)z \quad z\in\Omega_2 \tag{43b}$$

where

$$\Delta_1 = \frac{2c_{44}^2}{c_{44}^1}\frac{[c_{44}^1(\varepsilon_{11}^1+\varepsilon_{11}^2)+e_{15}^1(e_{15}^1+e_{15}^2)]}{\Delta}, \quad \Delta_2 = \frac{2c_{44}^2}{\varepsilon_{11}^1}\frac{(e_{15}^1\varepsilon_{11}^2-\varepsilon_{11}^1 e_{15}^2)}{\Delta},$$

$$\Delta_3 = \frac{1}{\Delta}[(c_{44}^1-c_{44}^2)(\varepsilon_{11}^1+\varepsilon_{11}^2)+(e_{15}^1)^2-(e_{15}^2)^2], \quad \Delta_4 = \frac{2c_{44}^1}{\varepsilon_{11}^1}\frac{(e_{15}^1\varepsilon_{11}^2-\varepsilon_{11}^1 e_{15}^2)}{\Delta},$$

$$\Delta_5 = \frac{2\varepsilon_{11}^2}{c_{44}^1}\frac{(c_{44}^1 e_{15}^2-e_{15}^1 c_{44}^2)}{\Delta}, \quad \Delta_6 = \frac{2\varepsilon_{11}^2}{\varepsilon_{11}^1}\frac{[\varepsilon_{11}^1(c_{44}^1+c_{44}^2)+e_{15}^1(e_{15}^1+e_{15}^2)]}{\Delta},$$

$$\Delta_7 = \frac{2\varepsilon_{11}^1}{c_{44}^1}\frac{(c_{44}^1 e_{15}^2-e_{15}^1 c_{44}^2)}{\Delta}, \quad \Delta_8 = \frac{1}{\Delta}[(c_{44}^1+c_{44}^2)(\varepsilon_{11}^1-\varepsilon_{11}^2)+(e_{15}^1)^2-(e_{15}^2)^2],$$

$$\Delta_9 = -\frac{2\varepsilon_{11}^2}{c_{44}^2}\frac{(c_{44}^1 e_{15}^2-e_{15}^1 c_{44}^2)}{\Delta}, \quad \Delta_{10} = \frac{2c_{44}^1}{c_{44}^2}\frac{[c_{44}^2(\varepsilon_{11}^1+\varepsilon_{11}^2)+e_{15}^2(e_{15}^1+e_{15}^2)]}{\Delta},$$

$$\Delta_{11} = -\frac{2\varepsilon_{11}^1}{c_{44}^2}\frac{(c_{44}^1 e_{15}^2-e_{15}^1 c_{44}^2)}{\Delta},$$

with

$$\Delta = (c_{44}^1 + c_{44}^2)(\varepsilon_{11}^1 + \varepsilon_{11}^2) + e_{15}^1 + e_{15}^2)^2.$$

It follows from (7), (42) and (43) that

$$E_{x1} - iE_{y1} = -\frac{1}{\varepsilon_{11}^1}\left[q_0 - (\Delta_7 \bar{p}_0 + \Delta_8 \bar{q}_0)\frac{a^2}{z^2} + \frac{c_{44}^2 b_z}{2\pi i}\Delta_{11}\left(\frac{1}{z - z_0} - \frac{1}{z}\right)\right] \tag{44a}$$

$$D_{x1} - iD_{y1} = \left(\frac{e_{15}^1}{c_{44}^1}p_0 - q_0\right) + \frac{e_{15}^1 b_z}{2\pi i}\frac{1}{z} + \frac{c_{44}^2 b_z}{2\pi i}\left(\frac{e_{15}^1}{c_{44}^1}\Delta_{10} - \Delta_{11}\right)\left(\frac{1}{z - z_0} - \frac{1}{z}\right)$$
$$- \left[\frac{e_{15}^1}{c_{44}^1}(\Delta_3 \bar{p}_0 + \Delta_4 \bar{q}) - (\Delta_7 \bar{p}_0 + \Delta_8 \bar{q}_0)\right]\frac{a^2}{z^2} \tag{44b}$$

$$\sigma_{zx1} - i\sigma_{zy1} = \left(p_0 + \frac{e_{15}^1}{\varepsilon_{11}^1}q_0\right) + \frac{c_{44}^1 b_z}{2\pi i}\frac{1}{z} + \frac{c_{44}^2 b_z}{2\pi i}\left(\Delta_{10} + \frac{e_{15}^1}{\varepsilon_{11}^1}\Delta_{11}\right)\left(\frac{1}{z - z_0} - \frac{1}{z}\right)$$
$$- \left[(\Delta_3 \bar{p}_0 + \Delta_4 \bar{q}_0) + \frac{e_{15}^1}{\varepsilon_{11}^1}(\Delta_7 \bar{p}_0 + \Delta_8 \bar{q}_0)\right]\frac{a^2}{z^2} \tag{44c}$$

in the matrix, and

$$E_{x2} - iE_{y2} = -\frac{1}{\varepsilon_{11}^2}\left[(\Delta_4 p_0 + \Delta_6 q_0) - \frac{c_{44}^2 b_z}{2\pi i}\frac{\Delta_9}{z - a^2/\bar{z}_0}\right] \tag{45a}$$

$$D_{x2} - iD_{y2} = \frac{e_{15}^2}{c_{44}^2}(\Delta_1 p_0 + \Delta_2 q_0) - (\Delta_5 p_0 + \Delta_6 q_0) + \frac{e_{15}^2 b_z}{2\pi i}\frac{2}{z - z_0}$$
$$+ \frac{c_{44}^2 b_z}{2\pi i}\left(\frac{e_{15}^2}{c_{44}^2}\Delta_3 + \Delta_9\right)\frac{1}{z - a^2/\bar{z}_0} \tag{45b}$$

$$\sigma_{zx2} - i\sigma_{zy2} = (\Delta_1 p_0 + \Delta_2 q_0) + \frac{e_{15}^2}{\varepsilon_{11}^2}(\Delta_5 p_0 + \Delta_6 q_0) + \frac{c_{44}^2 b_z}{2\pi i}\frac{1}{z - z_0}$$
$$+ \frac{c_{44}^2 b_z}{2\pi i}\left(\Delta_3 - \frac{e_{15}^2}{\varepsilon_{11}^2}\Delta_9\right)\frac{1}{z - a^2/\bar{z}_0} \tag{45c}$$

in the inhomogeneity.

Expressions (44) and (45) reveal that the existence of the dislocation will affect the electric field strength, electric displacements and stresses, both inside and outside the circular inhomogeneity. In the absence of the dislocation, these field components become uniform in the inhomogeneity and our results are identical to those obtained previously by Pak (1992). In the absence of the electric fields, our solution coincides with those of Smith (1968) and Gong and Meguid (1994) for the elastic inhomogeneity problem.

In the absence of p_0 and q_0, the interaction energy between a dislocation lying at the point $z_0 = x_0$ in the inhomogeneity and the force on the dislocation can be calculated from (40) and (41) as

$$\delta W = \frac{c_{44}^2 b_z^2}{4\pi}(\alpha_2 \Delta_9 - \Delta_3) \ln\left(1 - \frac{x_0^2}{a^2}\right) \tag{46}$$

and

$$F = -\frac{\partial(\delta W)}{x_0} = \frac{c_{44}^2 b_z^2}{2\pi} \frac{(\alpha_2 \Delta_9 - \Delta_3) x_0}{a^2 - x_0^2}, \tag{47}$$

respectively. These results reduce to those of Dundurs (1967), when only the elastic field is considered.

5.2. *Elliptical elastic dielectric inhomogeneity in elastic dielectric matrix*

If the screw dislocation is located inside an elliptical elastic dielectric inhomogeneity which is embedded in an elastic dielectric matrix, then $e_{15}^1 = e_{15}^2 = 0$. In this case, the expressions provided by (31) for the coefficients a_k^2, b_k^1, c_k^2 and d_k^1 can be given by the following simple forms

$$a_k^2 = I_k^{(1)} a_k^0 + J_k^{(1)} \bar{a}_k^0 + U_k^{(1)} b_k^* + V_k^{(1)} \bar{b}_k^*, \quad c_k^2 = L_k^{(3)} c_k^0 + N_k^{(3)} \bar{c}_k^0,$$

$$b_k^1 = I_k^{(2)} a_k^0 + J_k^{(2)} \bar{a}_k^0 + U_k^{(2)} b_k^* + V_k^{(2)} \bar{b}_k^* - b_k^0, \quad d_k^1 = L_k^{(4)} c_k^0 + N_k^{(4)} \bar{c}_k^0 - d_k^0 \tag{48}$$

where

$$I_k^{(1)} = \frac{2\mu_1(1+\mu_1)}{(1+\mu_1)^2 - (1-\mu_1)^2 R^{-4(k+1)}}, \quad J_k^{(1)} = -\frac{2\mu_1(1-\mu_1) R^{-2(k+1)}}{(1+\mu_1)^2 - (1-\mu_1)^2 R^{-4(k+1)}}$$

$$U_k^{(1)} = \frac{(1-\mu_1)^2 R^{-2(k+1)}}{(1+\mu_1)^2 - (1-\mu_1)^2 R^{-4(k+1)}}, \quad V_k^{(1)} = -\frac{1-\mu_1^2}{(1+\mu_1)^2 - (1-\mu_1)^2 R^{-4(k+1)}}$$

$$I_k^{(2)} = \frac{4\mu_1 R^{-2(k+1)}}{(1+\mu_1)^2 - (1-\mu_1)^2 R^{-4(k+1)}}, \quad J_k^{(2)} = -\frac{(1-\mu_1^2)[1 - R^{-4(k+1)}]}{(1+\mu_1)^2 - (1-\mu_1)^2 R^{-4(k+1)}}$$

$$U_k^{(2)} = \frac{2(1+\mu_1)}{(1+\mu_1)^2 - (1-\mu_1)^2 R^{-4(k+1)}}, \quad V_k^{(2)} = -\frac{2(1-\mu_1) R^{-2(k+1)}}{(1+\mu_1)^2 - (1-\mu_1)^2 R^{-4(k+1)}}$$

$$L_k^{(3)} = \frac{2\mu_2(1+\mu_2)}{(1+\mu_2)^2 - (1-\mu_2)^2 R^{-4(k+1)}}, \quad N_k^{(3)} = -\frac{2\mu_2(1-\mu_2) R^{-2(k+1)}}{(1+\mu_2)^2 - (1-\mu_2)^2 R^{-4(k+1)}}$$

$$L_k^{(4)} = \frac{4\mu_2 R^{-2(k+1)}}{(1+\mu_2)^2 - (1-\mu_2)^2 R^{-4(k+1)}}, \quad N_k^{(4)} = \frac{(1-\mu_2^2)[1 - R^{-4(k+1)}]}{(1+\mu_2)^2 - (1-\mu_2)^2 R^{-4(k+1)}}.$$

The field potentials are thus obtained as follows:

$$\Psi_0(\zeta) + \Psi_1(\zeta) = \frac{c_{44}^1 b_z}{2\pi i} \ln \zeta + \frac{c}{2} p_0 R \zeta + \frac{c}{2}(I_0^{(2)} p_0 + J_0^{(2)} \bar{p}_0) \frac{R}{\zeta}$$
$$+ \sum_{k=0}^{\infty} (U_k^{(2)} b_k^* + V_k^{(2)} \bar{b}_k^*) \zeta^{-(k+1)} \quad \zeta \in \Omega_1 \tag{49a}$$

$$\Phi_0(\zeta)+\Phi_1(\zeta)=\frac{c}{2}q_0R\zeta+\frac{c}{2}(L_0^{(4)}q_0+N_0^{(4)}\bar{q}_0)\frac{R}{\zeta}\quad \zeta\in\Omega_1 \tag{49b}$$

$$\Psi_*(\zeta)+\Psi_2(\zeta)=\frac{c_{44}^2b_z}{2\pi i}\ln(z-z_0)+(I_0^{(1)}p_0+J_0^{(1)}\bar{p}_0)z + \sum_{k=0}^{\infty}(U_k^{(1)}b_k^*+V_k^{(1)}\bar{b}_k^*)[\zeta^{k+1}+(R^2\zeta)^{-(k+1)}]\quad \zeta\in\Omega_2 \tag{50a}$$

$$\Phi_*(\zeta)+\Phi_2(\zeta)=(L_0^{(3)}q_0+N_0^{(3)}\bar{q}_0)z\quad \zeta\in\Omega_2 \tag{50b}$$

where b_k^* is given by (24e). If the dislocation is located at point $z_0=\Omega_0(\zeta_0)$ along the x-axis and the remote strain γ_{zy}^{∞} or stress σ_{zy}^{∞} vanishes, such that $\zeta_0=\bar{\zeta}_0$, $b_k^*=-\bar{b}_k^*$ and $p_0=\bar{p}_0$, then the field components can be given by

$$E_{x1}-iE_{y1}=-\frac{1}{\varepsilon_{11}^1}[q_0\zeta^2-(L_0^{(4)}q_0+N_0^{(4)}\bar{q}_0)]\frac{R^2}{R^2\zeta^2-1} \tag{51a}$$

$$D_{x1}-iD_{y1}=-[q_0\zeta^2-(L_0^{(4)}q_0+N_0^{(4)}\bar{q}_0)]\frac{R^2}{R^2\zeta^2-1} \tag{51b}$$

$$\sigma_{zx1}-i\sigma_{zy1}=\left\{\frac{c_{44}^1b_z}{\pi i}\zeta+p_0cR\zeta^2-p_0cR(I_0^{(2)}+J_0^{(2)}) -2\sum_{k=0}^{\infty}(k+1)b_k^*(U_k^{(2)}-V_k^{(2)})\zeta^{-k}\right\}\frac{R}{c(R^2\zeta^2-1)} \tag{51c}$$

in the matrix, and

$$E_{x2}-iE_{y2}=-\frac{1}{\varepsilon_{11}^2}(L_0^{(3)}q_0+N_0^{(3)}\bar{q}_0) \tag{52a}$$

$$D_{x2}-iD_{y2}=-(L_0^{(3)}q_0+N_0^{(3)}\bar{q}_0) \tag{52b}$$

$$\sigma_{zx2}-i\sigma_{zy2}=\frac{c_{44}^2b_z}{2\pi i}\frac{1}{z-z_0}+p_0(I_0^{(1)}+J_0^{(1)}) +\frac{2R\zeta^2}{c(R^2\zeta^2-1)}\sum_{k=0}^{\infty}\frac{(k+1)b_k^*}{R^k}(U_k^{(1)}-V_k^{(1)})[(R\zeta)^k-(R\zeta)^{-(k+2)}] \tag{52c}$$

in the inhomogeneity.

In this case, the elastic and the electric fields are decoupled. Accordingly, the electric field strength and the electric displacements are uniform in the inhomogeneity.

5.3. *Elliptical piezoelectric inhomogeneity in elastic matrix*

When the matrix becomes elastic and is subjected only to remote mechanical stresses σ_{zx}^{∞} and σ_{zy}^{∞} or remote mechanical strains γ_{zx}^{∞} and γ_{zy}^{∞}, then $e_{15}^{1} = \varepsilon_{11}^{1} = 0$ and $q_0 = 0$. Piezoelectric composite sensors are usually made in this configuration, where a piezoelectric bar is embedded in an elastic matrix. The field potentials in this case are given as follows

$$\Psi_0(\zeta)+\Psi_1(\zeta) = \frac{c_{44}^{1}b_z}{2\pi i}\ln\zeta+\frac{c}{2}p_0R\zeta+\frac{c}{2}(I_0^{(2)}p_0+J_0^{(2)}\bar{p}_0)\frac{R}{\zeta} + \sum_{k=0}^{\infty}(U_k^{(2)}b_k^*+V_k^{(2)}\bar{b}_k^*)\zeta^{-(k+1)} \quad \zeta\in\Omega_1 \tag{53a}$$

$$\Phi_0(\zeta)+\Phi_1(\zeta) = 0 \quad \zeta\in\Omega_1 \tag{53b}$$

$$\Phi_*(\zeta)+\Psi_2(\zeta) = \frac{c_{44}^{2}b_z}{2\pi i}\ln(z-z_0)+(I_0^{(1)}p_0+J_0^{(1)}\bar{p}_0)z + \sum_{k=0}^{\infty}(U_k^{(1)}b_k^*+V_k^{(1)}\bar{b}_k^*)[\zeta^{k+1}+(R^2\zeta)^{-(k+1)}] \quad \zeta\in\Omega_2 \tag{54a}$$

$$\Phi_*(\zeta)+\Phi_2(\zeta) = (I_0^{(3)}p_0+J_0^{(3)}\bar{p}_0)z+\sum_{k=0}^{\infty}(U_k^{(3)}b_k^*+V_k^{(3)}\bar{b}_k^*)[\zeta^{k+1}+(R^2\zeta)^{-(k+1)}] \quad \zeta\in\Omega_2 \tag{54b}$$

where

$$I_k^{(1)} = \frac{2\mu_1(1+\mu_1+\mu_1\alpha_2\beta_2)}{\omega}, \quad J_k^{(1)} = -\frac{2\mu_1(1-\mu_1-\mu_1\alpha_2\beta_2)R^{-2(k+1)}}{\omega};$$

$$U_k^{(1)} = \frac{(1-\mu_1-\mu_1\alpha_2\beta_2)^2R^{-2(k+1)}}{\omega}, \quad V_k^{(1)} = -\frac{1-\mu_1^2(1+\alpha_2\beta_2)^2}{\omega};$$

$$I_k^{(2)} = \frac{4\mu_1(1+\alpha_2\beta_2)R^{-2(k+1)}}{\omega}, \quad J_k^{(2)} = \frac{[1-\mu_1^2(1+\alpha_2\beta_2)^2][1-R^{-4(k+1)}]}{\omega};$$

$$U_k^{(2)} = \frac{2(1+\alpha_2\beta_2)(1+\mu_1+\mu_1\alpha_2\beta_2)}{\omega},$$

$$V_k^{(2)} = -\frac{2(1+\alpha_2\beta_2)(1-\mu_1-\mu_1\alpha_2\beta_2)R^{-2(k+1)}}{\omega};$$

$$I_k^{(3)} = \frac{2\mu_1\beta_2(1+\mu_1+\mu_1\alpha_2\beta_2}{\omega}, \quad J_k^{(3)} = -\frac{2\mu_1\beta_2(1-\mu_1-\mu_1\alpha_2\beta_2)R^{-2(k+1)})}{\omega}$$

$$U_k^{(3)} = \frac{-4\mu_1\beta_2(1+\alpha_2\beta_2)R^{-2(k+1)}}{[1-R^{-4(k+1)}]\omega},$$

$$V_k^{(3)} = \frac{-2\beta_2\{[1-R^{-4(k+1)}]+\mu_1(1+\alpha_2\beta_2)[1+R^{-4(k+1)}]\}}{[1-R^{-4(k+1)}]\omega}$$

with

$$\omega = (1+\mu_1+\mu_1\alpha_2\beta_2)^2-(1-\mu_1-\mu_1\alpha_2\beta_2)^2R^{-4(k+1)}.$$

The field components can be derived by substituting (53) and (54) into (7). It should be pointed out that due to the electric–elastic coupling, both the electric fields and the mechanical fields are influenced by the dislocation, and thus are not uniform inside the inhomogeneity.

6. Conclusions

A general treatment is provided to the electro-elastic interaction problem of a screw dislocation inside an elliptical piezoelectric inhomogeneity in an infinite piezoelectric matrix. By using conformal mapping and the perturbation method, explicit forms of the field potentials and the field components are derived in both the inhomogeneity and the matrix. The expressions for the internal energy of a dislocation inside the inhomogeneity and the force on the dislocation are given. Several particular problems are provided and are used not only to verify the validity of the current results, but also to determine the electro-mechanical coupling effects resulting from the presence of a point defect.

Appendix 1: Expressions for complex constants p_0 and q_0 corresponding to different combinations of remote electric and mechanical loads

The complex constants p_0 and q_0 in (19) can be determined from the following four cases of the boundary conditions given at infinity:

Case 1: Remote mechanical strains γ_{zx}^∞, γ_{zy}^∞ and remote electric field strength E_x^∞ and E_y^∞ will yield

$$p_0 = c_{44}^1\gamma_{zx}^\infty - ic_{44}^1\gamma_{zy}^\infty, \quad q_0 = -\varepsilon_{11}^1E_x^\infty + i\varepsilon_{11}^1E_y^\infty. \tag{A1.1}$$

Case 2: Remote mechanical stresses σ_{zx}^∞, σ_{zy}^∞ and remote electric displacements D_x^∞ and D_y^∞ will yield

$$p_0 = \frac{\sigma_{zx}^\infty+(e_{15}^1/c_{44}^1)D_x^\infty}{1+(e_{15}^1)^2/(\varepsilon_{11}^1c_{44}^1)} - i\frac{\sigma_{zy}^\infty+(e_{15}^1/c_{44}^1)D_y^\infty}{1+(e_{15}^1)^2/(\varepsilon_{11}^1c_{44}^1)},$$

$$q_0 = \frac{(e_{15}^1/c_{44}^1)\sigma_{zx}^\infty-D_x^\infty}{1+(e_{15}^1)^2/(\varepsilon_{11}^1c_{44}^1)} - i\frac{(e_{15}^1/c_{44}^1)\sigma_{zy}^\infty-D_y^\infty}{1+(e_{15}^1)^2/(\varepsilon_{11}^1c_{44}^1)}. \tag{A1.2}$$

Case 3: Remote mechanical strains γ_{zx}^{∞}, γ_{zy}^{∞} and remote electric displacements D_x^{∞} and D_y^{∞} will yield

$$p_0 = c_{44}^1\gamma_{zx}^{\infty} - ic_{44}^1\gamma_{zy}^{\infty}, \quad q_0 = (e_{15}^1\gamma_{zx}^{\infty} - D_x^{\infty}) - i(e_{15}^1\gamma_{zy}^{\infty} - D_y^{\infty}). \tag{A1.3}$$

Case 4: Remote mechanical stresses σ_{zx}^{∞}, σ_{zy}^{∞} and remote electric field strength E_x^{∞} and E_y^{∞} will yield

$$p_0 = (\sigma_{zx}^{\infty} + e_{15}^1 E_x^{\infty}) - i(\sigma_{zy}^{\infty} + e_{15}^1 E_y^{\infty}), \quad q_0 = -\varepsilon_{11}^1 E_x^{\infty} + i\varepsilon_{11}^1 E_y^{\infty}. \tag{A1.4}$$

Appendix 2: Details of coefficients in eqns (31)

The coefficients in eqns (31) are given as follows:

$$I_k^{(1)} = R^{2(k+1)}\left(\frac{\lambda_{1,k}}{\delta_{1,k}} + \frac{\lambda_{3,k}}{\delta_{2,k}}\right), \quad J_k^{(1)} = R^{2(k+1)}\left(\frac{\lambda_{1,k}}{\delta_{1,k}} - \frac{\lambda_{3,k}}{\delta_{2,k}}\right),$$

$$L_k^{(1)} = R^{2(k+1)}\left(\frac{\lambda_{2,k}}{\delta_{1,k}} + \frac{\lambda_{4,k}}{\delta_{2,k}}\right), \quad N_k^{(1)} = R^{2(k+1)}\left(\frac{\lambda_{2,k}}{\delta_{1,k}} - \frac{\lambda_{4,k}}{\delta_{2,k}}\right),$$

$$U_k^{(1)} = R^{2(k+1)}\left(\frac{\lambda_{9,k}}{\delta_{1,k}} - \frac{\lambda_{11,k}}{\delta_{2,k}}\right), \quad V_k^{(1)} = R^{2(k+1)}\left(\frac{\lambda_{9,k}}{\delta_{1,k}} + \frac{\lambda_{11,k}}{\delta_{2,k}}\right). \tag{A2.1}$$

$$I_k^{(3)} = R^{2(k+1)}\left(\frac{\lambda_{5,k}}{\delta_{1,k}} + \frac{\lambda_{7,k}}{\delta_{2,k}}\right), \quad J_k^{(3)} = R^{2(k+1)}\left(\frac{\lambda_{5,k}}{\delta_{1,k}} - \frac{\lambda_{7,k}}{\delta_{2,k}}\right),$$

$$L_k^{(3)} = R^{2(k+1)}\left(\frac{\lambda_{6,k}}{\delta_{1,k}} + \frac{\lambda_{8,k}}{\delta_{2,k}}\right), \quad N_k^{(3)} = R^{2(k+1)}\left(\frac{\lambda_{6,k}}{\delta_{1,k}} - \frac{\lambda_{8,k}}{\delta_{2,k}}\right),$$

$$U_k^{(3)} = R^{2(k+1)}\left(\frac{\lambda_{10,k}}{\delta_{1,k}} - \frac{\lambda_{12,k}}{\delta_{2,k}}\right), \quad V_k^{(3)} = R^{2(k+1)}\left(\frac{\lambda_{10,k}}{\delta_{1,k}} + \frac{\lambda_{12,k}}{\delta_{2,k}}\right). \tag{A2.2}$$

$$I_k^{(2)} = \frac{1+R^{2(k+1)}}{\mu_1}\frac{\lambda_{1,k}}{\delta_{1,k}} + \frac{1-R^{2(k+1)}}{\mu_1}\frac{\lambda_{3,k}}{\delta_{2,k}}, \quad J_k^{(2)} = \frac{1+R^{2(k+1)}}{\mu_1}\frac{\lambda_{1,k}}{\delta_{1,k}} - \frac{1-R^{2(k+1)}}{\mu_1}\frac{\lambda_{3,k}}{\delta_{2,k}} - 1,$$

$$L_k^{(2)} = \frac{1+R^{2(k+1)}}{\mu_1}\frac{\lambda_{2,k}}{\delta_{1,k}} + \frac{1-R^{2(k+1)}}{\mu_1}\frac{\lambda_{4,k}}{\delta_{2,k}}, \quad N_k^{(2)} = \frac{1+R^{2(k+1)}}{\mu_1}\frac{\lambda_{2,k}}{\delta_{1,k}} - \frac{1-R^{2(k+1)}}{\mu_1}\frac{\lambda_{4,k}}{\delta_{2,k}},$$

$$U_k^{(2)} = \frac{1+R^{2(k+1)}}{\mu_1}\frac{\lambda_{9,k}}{\delta_{1,k}} - \frac{1-R^{2(k+1)}}{\mu_1}\frac{\lambda_{11,k}}{\delta_{2,k}} + \frac{1}{\mu_1}, \quad V_k^{(2)} = \frac{1+R^{2(k+1)}}{\mu_1}\frac{\lambda_{9,k}}{\delta_{1,k}} + \frac{1-R^{2(k+1)}}{\mu_1}\frac{\lambda_{11,k}}{\delta_{2,k}}. \tag{A2.3}$$

$$I_k^{(4)} = \frac{1+R^{2(k+1)}}{\mu_2}\frac{\lambda_{5,k}}{\delta_{1,k}} + \frac{1-R^{2(k+1)}}{\mu_2}\frac{\lambda_{7,k}}{\delta_{2,k}}, \quad J_k^{(4)} = \frac{1+R^{2(k+1)}}{\mu_2}\frac{\lambda_{5,k}}{\delta_{1,k}} - \frac{1-R^{2(k+1)}}{\mu_2}\frac{\lambda_{7,k}}{\delta_{2,k}},$$

$$L_k^{(4)} = \frac{1+R^{2(k+1)}}{\mu_2}\frac{\lambda_{6,k}}{\delta_{1,k}} + \frac{1-R^{2(k+1)}}{\mu_2}\frac{\lambda_{8,k}}{\delta_{2,k}}, \quad N_k^{(4)} = \frac{1+R^{2(k+1)}}{\mu_2}\frac{\lambda_{6,k}}{\delta_{1,k}} - \frac{1-R^{2(k+1)}}{\mu_2}\frac{\lambda_{8,k}}{\delta_{2,k}} - 1,$$

$$U_k^{(4)} = \frac{1+R^{2(k+1)}}{\mu_2}\frac{\lambda_{10,k}}{\delta_{1,k}} - \frac{1-R^{2(k+1)}}{\mu_2}\frac{\lambda_{12,k}}{\delta_{2,k}}, \quad V_k^{(4)} = \frac{1+R^{2(k+1)}}{\mu_2}\frac{\lambda_{10,k}}{\delta_{1,k}} + \frac{1-R^{2(k+1)}}{\mu_2}\frac{\lambda_{12,k}}{\delta_{2,k}}. \tag{A2.4}$$

where

$$\lambda_{1,k} = -\left[\left(\frac{1}{\mu_2}+1\right)R^{2(k+1)} + \left(\frac{1}{\mu_2}-1\right)\right] - \beta_1\left[\left(\frac{\alpha_1}{\mu_2}+\alpha_2\right)R^{2(k+1)} + \left(\frac{\alpha_1}{\mu_2}-\alpha_2\right)\right] \tag{A2.5}$$

$$\lambda_{2,k} = (\alpha_1-\alpha_2)(1-R^{2(k+1)}) \tag{A2.6}$$

$$\lambda_{3,k} = -\left[\left(\frac{1}{\mu_2}+1\right)R^{2(k+1)} - \left(\frac{1}{\mu_2}-1\right)\right] - \beta_1\left[\left(\frac{\alpha_1}{\mu_2}+\alpha_2\right)R^{2(k+1)} - \left(\frac{\alpha_1}{\mu_2}-\alpha_2\right)\right] \tag{A2.7}$$

$$\lambda_{4,k} = (\alpha_2-\alpha_1)(1+R^{2(k+1)}) \tag{A2.8}$$

$$\lambda_{5,k} = (\beta_2-\beta_1)(1-R^{2(k+1)}) \tag{A2.9}$$

$$\lambda_{6,k} = -\left[\left(\frac{1}{\mu_1}+1\right)R^{2(k+1)} + \left(\frac{1}{\mu_1}-1\right)\right] - \alpha_1\left[\left(\frac{\beta_1}{\mu_1}+\beta_2\right)R^{2(k+1)} + \left(\frac{\beta_1}{\mu_1}-\beta_2\right)\right] \tag{A2.10}$$

$$\lambda_{7,k} = (\beta_1-\beta_2)(1+R^{2(k+1)}) \tag{A2.11}$$

$$\lambda_{8,k} = -\left[\left(\frac{1}{\mu_1}+1\right)R^{2(k+1)} - \left(\frac{1}{\mu_1}-1\right)\right] - \alpha_1\left[\left(\frac{\beta_1}{\mu_1}+\beta_2\right)R^{2(k+1)} - \left(\frac{\beta_1}{\mu_1}-\beta_2\right)\right] \tag{A2.12}$$

$$\lambda_{9,k} = \frac{1}{2}\left(\frac{1}{\mu_1}-1\right)\left[\left(\frac{1}{\mu_2}+1\right)R^{2(k+1)} + \left(\frac{1}{\mu_2}-1\right)\right] + \frac{1}{2}\left(\frac{\beta_1}{\mu_1}-\beta_2\right)\left[\left(\frac{\alpha_1}{\mu_2}+\alpha_2\right)R^{2(k+1)} + \left(\frac{\alpha_1}{\mu_2}-\alpha_2\right)\right] \tag{A2.13}$$

$$\lambda_{10,k} = (\beta_2-\beta_1)R^{2(k+1)}/\mu_1 \tag{A2.14}$$

$$\lambda_{11,k} = \frac{1}{2}\left(\frac{1}{\mu_1}-1\right)\left[\left(\frac{1}{\mu_2}+1\right)R^{2(k+1)} - \left(\frac{1}{\mu_2}-1\right)\right] + \frac{1}{2}\left(\frac{\beta_1}{\mu_1}-\beta_2\right)\left[\left(\frac{\alpha_1}{\mu_2}+\alpha_2\right)R^{2(k+1)} - \left(\frac{\alpha_1}{\mu_2}-\alpha_2\right)\right] \tag{A2.15}$$

$$\lambda_{12,k} = (\beta_2 - \beta_1) R^{2(k+1)}/\mu_1 \tag{A2.16}$$

with

$$\delta_{1,k} = -\left[\left(\frac{1}{\mu_1}+1\right)R^{2(k+1)}+\left(\frac{1}{\mu_1}-1\right)\right]\left[\left(\frac{1}{\mu_2}+1\right)R^{2(k+1)}+\left(\frac{1}{\mu_2}-1\right)\right]$$
$$-\left[\left(\frac{\beta_1}{\mu_1}+\beta_2\right)R^{2(k+1)}+\left(\frac{\beta_1}{\mu_1}-\beta_2\right)\right]\left[\left(\frac{\alpha_1}{\mu_2}+\alpha_2\right)R^{2(k+1)}+\left(\frac{\alpha_1}{\mu_2}-\alpha_2\right)\right] \tag{A2.17}$$

$$\delta_{2,k} = -\left[\left(\frac{1}{\mu_1}+1\right)R^{2(k+1)}-\left(\frac{1}{\mu_1}-1\right)\right]\left[\left(\frac{1}{\mu_2}+1\right)R^{2(k+1)}+\left(\frac{1}{\mu_2}-1\right)\right]$$
$$-\left[\left(\frac{\beta_1}{\mu_1}+\beta_2\right)R^{2(k+1)}-\left(\frac{\beta_1}{\mu_1}-\beta_2\right)\right]\left[\left(\frac{\alpha_1}{\mu_2}+\alpha_2\right)R^{2(k+1)}-\left(\frac{\alpha_1}{\mu_2}-\alpha_2\right)\right] \tag{A2.18}$$

and μ_1, μ_2, α_1, α_2, β_1 and β_2 given in (17).

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